

Quantum Galilean Cannon as a Schrödinger Cat

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A quantum Galilean cannon is a 1D sequence of N hard-core particles with special mass ratios, and a hard wall; conservation laws due to the reflection group A_N prevent both classical stochastization and quantum diffraction. It is realizable through specie-alternating mutually repulsive bosonic soliton trains. We show that an initial disentangled state can evolve into one where the heavy and light particles are entangled, and propose a sensor, containing N_a atoms, with a $\sqrt{N_a}$ times higher sensitivity than in a one-atom sensor with N_a repetitions.

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In a 1D system of hard-core particles, by tuning the ratios between the particle masses one can choose a variety of distinct regimes of motion [1, 2]. While generic values result in thermalization, for some special ones the system maps to known multidimensional kaleidoscopes [3, 4]. The outcome velocities then become determined by the initial velocities only, independent of the initial positions if the particle ordering is preserved. In the quantum version, the eigenstates are finite superpositions of plane waves, with no diffraction [5]. In particular a (run in reverse) Galilean cannon—a hard wall followed by a 1D sequence of N hard-core particles with mass ratios of $1 : \frac{1}{3} : \frac{1}{6} : \frac{1}{10} : \dots : \frac{2}{N(N+1)}$ —will evolve from a state where only the lightest particle is moving (approaching the others from infinity), to a state where all of the particles are moving away from the wall, with the same speeds. The final state of the heavy particles is very different from the initial one, yet highly predictable, presenting an opportunity: if created, a superposition of the final and initial states would be a Schrödinger cat state [6]: just as the α -particle controls the well-being of the cat, so the state of the lightest particle controls either (a) the rest of the particles or (b) the heaviest particle, after the others are detected. And yet the whole system is in a pure state. The stored entanglement stored is ready to be used, and we will propose an interferometric application.

Consider a system of N 1D hard-core particles with masses m_1, m_2, \dots, m_N on the half-line $x > 0$, bounded by a hard wall at $x = 0$. The Hamiltonian is given by the kinetic energy,

$$\hat{H} = - \sum_{i=1}^N \frac{\hbar^2}{2m_i} \frac{\partial^2}{\partial x_i^2}; \quad (1)$$

the wavefunction $\Psi(x_1, x_2, \dots, x_N)$ satisfies the boundary conditions

$$\Psi|_{x_1=0} = \Psi|_{x_1=x_2} = \dots = \Psi|_{x_{N-1}=x_N} = 0. \quad (2)$$

The coordinate transformation $x_i = \sqrt{\mu/m_i} z_i$ for $i = 1, 2, \dots, N$, where μ is an arbitrary mass scale that can

be chosen at will, converts the system to a single N -dimensional particle of mass μ moving inside a mirror-walled wedge formed by N mirrors; the outward normalized normals to its mirrors are given by $\mathbf{n}_1 = -\mathbf{e}_1$ and $\mathbf{n}_i = \sqrt{m_i/(m_{i-1} + m_i)}\mathbf{e}_{i-1} - \sqrt{m_{i-1}/(m_{i-1} + m_i)}\mathbf{e}_i$ for $i = 2, 3, \dots, N$, where \mathbf{e}_i is the unit vector along the z_i -axis.

For a generic set of masses, sequential reflections about the mirrors generate an infinite set of spatial transformations. However, for every full reflection group [7] of a regular multidimensional polyhedron (i.e. Platonic solid), there is a set of masses whose corresponding system of mirrors generates that group [4]. In these cases, the eigenstates of the system can be found exactly through Bethe ansatz [3, 4]: they are given by finite linear combinations of plane waves. In particular, the “Galilean cannon” set of masses $m_1, m_2 = m_1/3, m_3 = m_1/6, \dots, m_N = \frac{2}{N(N+1)}m_1$ corresponds to the symmetry group of a regular N -dimensional tetrahedron [8].

As the initial state for the reverse Galilean cannon (Fig. 1), we take the tensor product of the Gaussian wavepackets for each particle, supplemented by its images needed to satisfy the boundary conditions (2). The corresponding solution of the time-dependent Schrödinger equation with the Hamiltonian (1), subject to the boundary conditions (2) is

$$\Psi(z_1, z_2, \dots, z_N) = \left(\sum_{\hat{g}} (-1)^{\mathcal{P}(\hat{g})} \hat{g} \right) \prod_{i=1}^N \psi(z_i, t | z_i^{(0)}, v_{z,i}, \sigma_{z,i}), \quad (3)$$

where

$$\begin{aligned} \psi(z, t | z^{(0)}, v_z, \sigma_z) \\ = A(t) e^{-\frac{(z-z^{(0)})^2 - 4i\hbar\mu\sigma_z^2 v_z(z-z^{(0)}) + 2i\mu t v_z^2 \sigma_z^2 / \hbar}{4\sigma_z^2 [1 + (i\hbar t)/(2\mu\sigma_z^2)]}} \end{aligned}$$

is the one-body Gaussian wavepacket expanding freely; $A(t) = (2\pi\sigma_z^2 [1 + (i\hbar t)/(2\mu\sigma_z^2)]^2)^{-1/4}$ is the normalization constant; the operators \hat{g} are the transformations

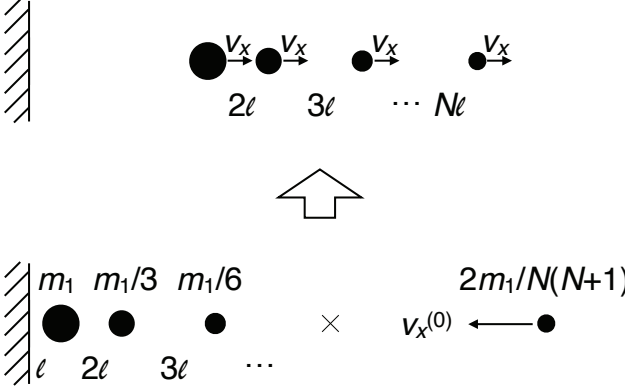


FIG. 1. Galilean cannon run in reverse: the lightest particle sets in motion all the heavy particles, at the same speed. In addition, the initial positions of the particles are chosen such a way that there exists a particle configuration—name the initial positions of all particles but the lightest, whose position is instead indicated by the cross—that is realized twice in course of the time evolution.

of space generated by the sequential application of reflections about the N mirrors given by the normals \mathbf{n}_i described above. This solution is obtained in the same way as the Bethe eigenstates in the case of hard-wall kaleidoscopes [4], which, in turn, is a generalization of a general solution for kaleidoscopes with Robin's boundary conditions [3, 9–14], which was inspired by the Bethe ansatz solutions for a gas of bosons [15–18]. The set \hat{g} 's forms the reflection group A_N , containing $(N + 1)$ elements [19]. $\mathcal{P}(\hat{g})$ is the parity of the group element, i.e. the parity of the number of reflections about the generating mirrors (given in particular by the normals \mathbf{n}_i above) needed to produce this element.

Having obtained the general solution for the problem, let us return to the physical coordinates x_1, x_2, \dots, x_N . For the sequence depicted in Fig. 1 (“the Galilean cannon run in reverse”), the initial velocities of the particles are $v_{x,N} = -\mathcal{V}_x^{(0)} < 0$ and $v_{x,1} = v_{x,2} = \dots = v_{x,N-1} = 0$. At the final stage of the process, each particle moves away from the wall with the same speed $\mathcal{V}_x = \sqrt{m_N/M} \mathcal{V}_x^{(0)} = \mathcal{V}_x^{(0)}/N$, where $M = \sum_{i=1}^N m_i$ is the total mass of the system. The initial distances between the particles, and between the leftmost particle and the wall, are assumed much greater than the widths of their initial packets: $x_1^{(0)} \gg \sigma_{x,1}$ and $x_i^{(0)} - x_{i-1}^{(0)} \gg \max(\sigma_{x,i}, \sigma_{x,i-1})$ for $i = 2, 3, \dots, N$. In this case, the initial state is close to a *product* state of individual non-overlapping states of finite support: the images, while formally present in the expression (3), will be exponentially small at $t = 0$ (but will come to prominence at later times, as the particle wavepackets move around and broaden).

If the initial multidimensional Gaussian wavepacket is sufficiently long, the various parts of the superposition

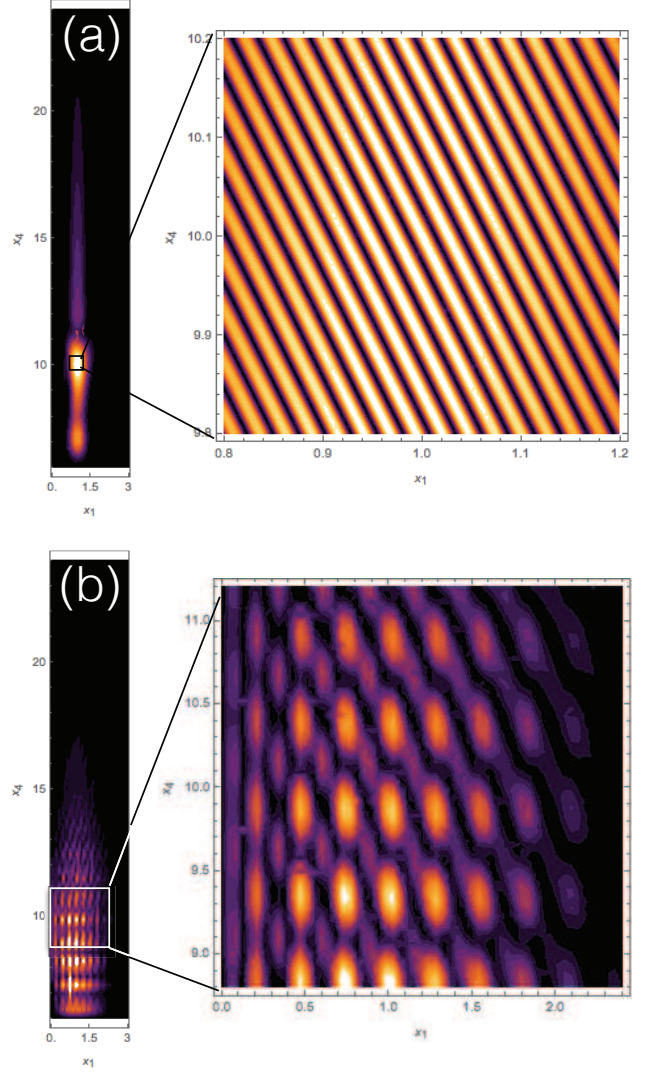


FIG. 2. A Schrödinger cat state produced in a Galilean cannon system with $N = 4$ particles. The conditional two-body density distribution for the heaviest and the lightest particles, $|\Psi(x_1, 3\ell, 6\ell, x_4)|^2$, is plotted, at a time t_{obs} , subject to the second and the third particles being held at $(x_2, x_3) = (3, 6)\ell$. The initial state is a Gaussian wavepacket for each of the particles. The mean initial velocities of all the particles but the lightest vanish, while the lightest is approaching the system at a speed $\mathcal{V}_x^{(0)}$. The magnified portions are centered around the self-crossing point, $(x_1, x_2, x_3, x_4) = (1, 3, 6, 10)\ell$, where the head of the distribution is capable of crossing its tail. In the figures, the head part is not visible outside of the crossing with the tail. The centers of the initial Gaussians are at $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = (1, 3, 6, 31.5)\ell$. In (a), the speed $\mathcal{V}_x^{(0)}$, the observation time, and the dispersions of the initial Gaussians are $\mathcal{V}_x^{(0)} = 949 \hbar/(m_1 \ell)$, $t_{\text{obs}} = 0.0332 m_1 \ell^2/\hbar$, and $(\sigma_{x,1}, \sigma_{x,2}, \sigma_{x,3}, \sigma_{x,4}) = (0.129, 0.223, 0.315, 9.49)\ell$ respectively. For (b), $\mathcal{V}_x^{(0)} = 94.9 \hbar/(m_1 \ell)$, $t_{\text{obs}} = 0.332 m_1 \ell^2/\hbar$, $(\sigma_{x,1}, \sigma_{x,2}, \sigma_{x,3}, \sigma_{x,4}) = (0.407, 0.705, 0.997, 9.49)\ell$. In both cases, the observation time corresponds to the exact middle point of the time evolution (see main text).

(3) will start overlapping at intermediate stages of the time evolution, forcing the particles to entangle—despite the closeness of the initial state to a product state. The most promising is the superposition between 1. the initial packet and 2. the “outgoing” one, where all the particles are moving with a velocity $+\mathcal{V}_x$: here the state of the lightest particle controls the state of each of the heavier ones, including the heaviest. It turns out that for a properly tuned set of the initial conditions, there will be regions of space where these two waves *spatially overlap* and all other parts of (3) are exponentially small. Indeed, one can show that if a classical trajectory passes through the point $(x_1^{(\text{sc})}, x_2^{(\text{sc})}, \dots, x_N^{(\text{sc})}) = (\ell, 3\ell, 6\ell, \dots, \frac{N(N+1)}{2}\ell)$, it will do so *twice*, once during the initial leg of the evolution and once during the final. (Here and below, ℓ is an arbitrary length scale, and the subscript ‘sc’ stands for “self-crossing.”) The distances between the particles increase linearly with the index: $x_j^{(\text{sc})} - x_{j-1}^{(\text{sc})} = j\ell$. At the exact middle point of the time evolution (which can be shown to equal the time when the lightest particle would hit the wall if there were no other particles present), the state around the point of self-crossing is close to

$$\Psi(x_1, x_2, \dots, x_N) \propto e^{-i(m_N \mathcal{V}_x^{(0)}(x_N - x_N^{(\text{sc})}) + \frac{\phi}{2})} \\ + e^{+i(\sum_{i=1}^{N-1} m_i \mathcal{V}_x(x_i - x_i^{(\text{sc})}) + m_N \mathcal{V}_x(x_N - x_N^{(\text{sc})}) + \frac{\phi}{2})},$$

where the relative phase ϕ can be approximated, using the eikonal approximation, as $\hbar\phi = 2m_1 \mathcal{V}_x^{(0)}\ell$. In the state above, the coordinate of the lightest particle is entangled with the center-of-mass position for the remaining bodies. Below, we will use this entanglement as a way to improve the sensitivity of interferometric measurements. As for the “cat” per se, we have the position of the center of mass of the particles being spread over an $\sim N^2\ell$ range. If this seems too abstract, one can also generate entanglement between two localized objects, one light and one heavy: suppose the intermediate particles 2, 3, \dots , $N-1$ have been detected at particular positions. The particles 1 and N remain entangled, in spite of the $N-2$ hard walls between them formed by the detected intermediate particles. For example, when the intermediate particles are detected at their “self-crossing” values, the state of the system becomes

$$\Psi_{1,N}(x_1, x_N) \propto e^{-i(m_N \mathcal{V}_x^{(0)}(x_N - x_N^{(\text{sc})}) + \frac{\phi}{2})} \\ + e^{+i(m_1 \mathcal{V}_x(x_1 - x_1^{(\text{sc})}) + m_N \mathcal{V}_x(x_N - x_N^{(\text{sc})}) + \frac{\phi}{2})}.$$

This state is a paradigmatic Schrödinger cat state where a light particle (x_N), the “ α -particle”, is entangled with a heavy one (x_1), the “cat”.

Figure 2 shows the results of time propagation according to the above scheme, for $N = 4$. The first set of parameters (Fig. 2(a)) assumes that the incident veloc-

ity of the light particle can be made as high as needed. In this case, the expansion of the initial wavepackets can be reduced to a point where at the self-crossing point, only the incident and the outgoing waves are allowed to be present. The clear interference fringes in the x_1 - x_4 plane are a signature of that. In the “worst case scenario” of Fig. 2(b), it is assumed that the available initial speed of the light particle is limited to a value ten times lower than the one in the first case. In this case, the particle packets, albeit non-overlapping in the beginning, spread and subsequently overlap at the detection instance. In particular, all two-body collision sequences will be explored. For a generic set of masses, the system would evolve towards an excited gas cloud interacting with a wall, with a resulting stochastization of the velocities. In the integrable case however, the evolution of the velocity distribution does *not* depend on the order in which the two-body collisions occur: this property reduces the variety of the wavevectors produced, and in doing so helps to preserve the x_1 - x_4 correlations. These correlations are still apparent from the Fig. 2(b).

We have numerically computed the Rényi entropy $S_2[\text{particle 1}] \equiv -\ln[\text{Tr}[\hat{\rho}_{\text{particle 1}}^2]]$ for the reduced density matrix $\hat{\rho}_{\text{particle 1}}$ of the heaviest particle, for the state truncated to the square area in both Fig. 2(a) and Fig. 2(b), and (x_2, x_3) fixed to $(3, 6)\ell$. To compute the entropy, we discretized the (x_1, x_4) space onto a square grid, and in doing so, reduced the computation to a standard setting where the Hilbert space has a finite number of dimensions. We verified that for small enough step of the grid, the entropy does not depend on the value of the step. As expected, in (a), $S_2[\text{particle 1}] \approx \ln(1.991)$: this value is close to $\ln(2)$ indicating two element-wise-distinct sets of particle momenta. Surprisingly, the case (b) also shows some appreciable entanglement, with $S_2[\text{particle 1}] \approx \ln(1.602)$. Naively, one would expect that in a state that correspond to wide spatial particle distributions, the correlation between the particles 1 and 4—separated by two distant “walls” represented by particles 2 and 3—will be lost. However, the integrability of the system reduces the number of available wavevectors to only a few, preserving the correlations. In (b), the contribution of the state of the particle 1 that multiplies the outgoing wave for the particle 4 is higher, due to the new waves appearing: the overall state of the particle 1 is “purer”, and the entropy is lower as the result.

Notice that, if plotted as a function of the coordinate $\tilde{X} \equiv X_{\text{COM}} + \frac{1}{N}x_N$, the N -body density corresponding to the state $\Psi(x_1, x_2, \dots, x_N)$ in the vicinity of the self-crossing point shows interference fringes with a distance between the crests of $\Delta\tilde{X} = 2\pi/(M\mathcal{V}_x)$, as if produced by a wave for a single massive particle of mass M split by a $\pm M\mathcal{V}_x/2$ beamsplitter. Here, $X_{\text{COM}} \equiv \sum_{i=1}^N m_i x_i / M$ is the center of mass coordinate. Figure 3 shows a sample interferometric scheme exploiting this effect. In this

scheme, the *beamsplitter only acts on the lightest particle*, while due to the entanglement buildup, the more massive particles also affect the position of the fringes. In what follows, we will regard the lightest particle as an individual atom: the heaviest one will become a polymer (a soliton, see below) with $N(N+1)/2 \sim N_a$ atoms, where N_a is the total number of atoms in the system. Imagine that a phase object, a potential barrier of height U , *per atom*, acting during a limited time τ , is introduced between the wall and the default position of the heavy polymer. The best sensitivity to the strength U of the phase object can be easily estimated as $\Delta U_{\text{Galilean cannon}} \sim \hbar/(\tau N_a)$, i.e. by the energy-time uncertainty relation for an interferometer formed by the heaviest atom attempting to measure a barrier of height $N_a U$ within a time τ . However, if only the individual atoms of mass m_N are used, the maximal sensitivity is only $\Delta U_{\text{individual atoms}} \sim \hbar/(\tau N_a^{1/2})$, which is the sensitivity of a single-atom interferometer further improved by the signal-to-noise reduction using N_a repetitive measurements. The net relative sensitivity gain produced by the entanglement, for a given number of atoms N_a available, becomes *gain* $\sim N_a^{1/2}$.

In conclusion, we showed that for a particular one-dimensional mass sequence, it is possible to realize a protocol in which the system evolves, on its own, from a product state to a state where a heavy particle becomes entangled with a light one, thus realizing Schrödinger's a-cat-and-an- α -particle paradigm. We show numerically that the Rényi entropy of the heavy particle can rise to almost $\ln(2)$. The robustness of the protocol is due to the integrability of the model that protects it from both classical stochastization and quantum diffraction. We suggest a concrete way to exploit the heavy-light entanglement by proposing an atomic interferometric sensor scheme that shows an $N_a^{1/2}$ increase in sensitivity.

As an empirical realization of the scheme presented above we suggest using chains of cold bosonic solitons [20–22]. For our scheme, it is necessary to have two internal states available (or, alternatively, two kinds of atoms). We assume that like species attract each other, while the scattering length between the opposite species is tuned to a positive value. The soliton sizes must be then adjusted to fit the desired mass sequence. The kinetic energy of the relative motion of the solitons must be lower than both the intra- and inter-specie interaction energy per particle, to ensure both a suppression of the inelastic effects and an absence of inter-specie transmission. Nontrivial integrals of motion present, at the mean-field level, in cold one-dimensional Bose gases may provide a way to both prevent the internal excitation of the solitons and to accurately divide the gas onto desired fractions [23]. For Li^7 atoms, the desired window in Feshbach magnetic field strength does exist: in particular, at 855 G, the scattering lengths governing a ($m_F = -1$)–

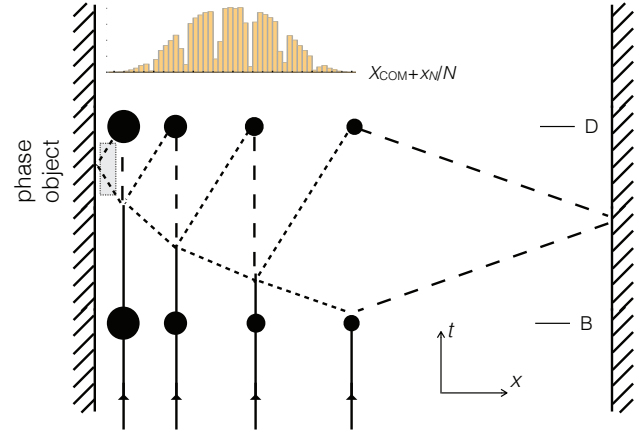


FIG. 3. A Galilean-cannon-based interferometer: an example with $N = 4$ particles. Initially all particles are at the point of self-crossing, $(x_1, x_2, x_3, x_4) = (1, 3, 6, \dots, \frac{N(N+1)}{2}) \frac{N=4}{10} \ell$. At time B, a beam-splitter is applied to the lightest particle. After a series of collisions, each particle returns to its initial position, at the detection time D, where the particle positions are measured. A phase object, both spatially and temporarily localized, is shown as a grey rectangle. While no individual particle position distribution possesses any structure, the distribution of the $\tilde{X} \equiv X_{\text{COM}} + \frac{1}{N}x_N$ variable shows interference fringes, whose position is controlled by the phase introduced by the object. The insert shows a sample histogram derived from 79900 simulated realizations of the detection cycle: those were selected from a longer sequence; only the realizations with particle positions within a hypercube of dimensions $0.1\ell \times 0.1\ell \times 0.1\ell \times 0.1\ell$ centered at the self-crossing point $(1, 3, 6, 10)\ell$ were kept. The distance between the crests is consistent with the predicted value of $\Delta\tilde{X} = 2\pi/(M\mathcal{V}_x) = 0.0165\ell$. The velocity kick induced by the beam-splitter is the same as the initial velocity $\mathcal{V}_x^{(0)}$ of Fig. 2(a). The initial wavepacket width of the lightest particle is $\sigma_{x,4} = 0.408\ell$. The period of time between the beamsplitting and detection is $0.0210 m_1 \ell^2/\hbar$. The remaining parameters are the same as in Fig. 2(a).

($m_F = 0$) mixture are $a_{-1,-1} \approx -0.5 a_B$, $a_{0,0} \lesssim -10 a_B$, and $a_{-1,0} \approx +1.0 a_B$, where a_B is the Bohr radius [24].

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